Analytical expressions for the coupling impedance of a long narrow slot in a coaxial beam pipe

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(Received 23 January 1998)

The longitudinal impedance and loss factor for a long narrow slot in a coaxial pipe are calculated by means of the modified Bethe's diffraction theory. The effects of the interference between the fields scattered by dipoles along the slot are taken into account, obtaining a final expression valid even for slots longer than the wavelength. [S1063-651X(98)14909-7]

PACS number(s): 41.75.−i, 41.20.−q, 29.27.−a

I. INTRODUCTION

Several recent papers have been devoted to the study of the effect, on the beam dynamics, of pumping holes and slots coupling the vacuum chamber to an external antichamber.

At low frequency, when the wavelength is larger than the aperture size, the problem is treated in terms of the static polarizabilities of the apertures and the typical quantities of interest, such as loss factor and coupling impedance, can be calculated by different methods [1,2].

At higher frequency, these procedures can be no longer followed. Recently frequency dependant polarizabilities have been introduced as presented in [3].

In this paper we analyze the case of a long slot in a coaxial vacuum chamber using a different method, where the slot is subdivided in infinitesimal slices [4] for which the static approximation is still valid. The problem is solved self-consistently in the framework of the modified Bethe's diffraction theory [5], which has been successfully applied to other cases [2,6,7].

In Sec. II we expose the basis of the theory leading to a set of integral equation. In Sec. III we find an approximate solution, and derive the analytical expressions for coupling impedance and loss factor. In Sec. IV we discuss the asymptotic behavior of the solution and in Sec. V the results are compared to numerical codes.

II. GENERAL THEORY

We consider a long and narrow slot on the inner tube of a coaxial beam pipe (Fig. 1). By applying the superposition principle, the slot can be thought of as composed of an infinite number of infinitesimal slices of length dz and width W [4].

According to the modified Bethe's diffraction theory, we may write the differential equivalent dipole moments for each slice as

$$dM_\varphi(z)=[H_{0\varphi}(z)-H_{s\varphi}(z)]d\alpha_m, \tag{1a}$$

$$dP_r(z)=\varepsilon[E_{0r}(z)-E_{sr}(z)]d\alpha_e, \tag{1b}$$

where $H_{0\varphi}(z)$ and $E_{0r}(z)$ are the primary fields generated by a point charge q, traveling with velocity c along the axis of a perfecting conducting pipe, while $H_{s\varphi}(z)$ and $E_{sr}(z)$ are the fields scattered by the slot. The coefficients of the scattered fields modal expansion are determined through the Lorentz reciprocity principle and they depend on the field sources, that is the dipole moments on each slice. The differential polarizabilities $d\alpha_m$ and $d\alpha_e$ are determined by averaging the static polarizabilities along the slot length [4]. Thus

$$d\alpha_m=\alpha_m\frac{dz}{L}, \tag{2a}$$

$$d\alpha_e=\alpha_e\frac{dz}{L}. \tag{2b}$$

For the sake of simplicity, we limit our analysis to frequencies below the cutoff of the TE_{11} mode in the inner and outer pipe. In this case Eqs. (1), as shown in the Appendix, become

$$\frac{dM_\varphi}{dz}(z)=\frac{\alpha_m}{L}\left[H_{0\varphi}(z)-j\frac{\omega}{2}\mu h_{0\varphi}^2\int_{\text{slot}}\frac{dM_\varphi}{d\xi}e^{-jk_0|z-\xi|}d\xi+j\frac{\omega}{2}h_{0\varphi}e_{0r}\int_{\text{slot}}\text{sgn}(\xi-z)\frac{dP_r}{d\xi}e^{-jk_0|z-\xi|}d\xi\right], \tag{3a}$$

$$\frac{dP_r}{dz}(z)=\frac{\varepsilon\alpha_e}{L}\left[E_{0r}(z)-j\frac{\omega}{2}e_{0r}^2\int_{\text{slot}}\frac{dP_r}{d\xi}e^{-jk_0|z-\xi|}d\xi+j\frac{\omega}{2}\mu h_{0\varphi}e_{0r}\int_{\text{slot}}\text{sgn}(\xi-z)\frac{dM_\varphi}{d\xi}e^{-jk_0|z-\xi|}d\xi\right]. \tag{3b}$$

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Equations (3) constitute a system of integral equations for dM_φ/dz and dP_r/dz. From a physical point of view, we may see that the scattered fields depend on the electric and magnetic dipoles all over the aperture, since each infinitesimal slice radiates a forward and a backward wave in the coaxial region. While the waves produced by the electric and magnetic dipoles of each single element are in phase along the beam direction, they are in phase opposition along the other.

Once system (3) has been solved, it is straightforward to derive the longitudinal coupling impedance [8]

$$Z(\omega)=-\frac{1}{q}\int_{-\infty}^{+\infty}E_z(r=0)e^{jk_0z}dz, \tag{4}$$

since the longitudinal electric field on the beam pipe axis is given by the $TM_{0,m}$ modes only [1]. Each slice contributes to the total impedance with

$$dZ(\omega)=j\,\frac{\omega Z_0}{2\pi qb}\left(\frac{1}{c}\,dM_\varphi+dP_r\right)e^{jk_0z}. \tag{5}$$

Summing up all the infinitesimal contributions we get

$$Z(\omega)=\int_{\text{slot}}dZ=j\,\frac{\omega Z_0}{2\pi qb}\int_{-L/2}^{L/2}\left(\frac{1}{c}\,\frac{dM_\varphi}{dz}+\frac{dP_r}{dz}\right)e^{jk_0z}dz. \tag{6}$$

The above expression can be seen as a generalization of the N holes interference problem in a coaxial beam pipe [7].

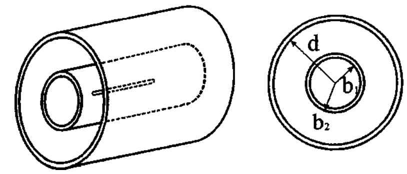

FIG. 1. Relevant geometry.

III. ANALYTICAL EXPRESSIONS FOR IMPEDANCE AND LOSS FACTOR

Integral equations (3) could be dealt with by means of the standard analytical or numerical techniques. However, since the energy radiated trough the hole is only a minor fraction of the total incident energy and the scattered field can be considered as a small perturbation with respect to the primary field, system (3) can be treated with perturbative procedures. The simplest approach is the iterative solution stopped at first order. In other words, we assume that in the integral expression of the right-hand side of Eqs. (3)

$$\left(\frac{dM_\varphi}{dz}\right)^{0\text{th}}=\frac{\alpha_m}{L}\,H_{0\varphi}(z), \tag{7a}$$

$$\left(\frac{dP_r}{dz}\right)^{0\text{th}}=\frac{\varepsilon\alpha_e}{L}\,E_{0r}(z). \tag{7b}$$

Substituting Eqs. (7) in Eqs. (3), we get the first order solution

$$\left(\frac{dM_\varphi}{dz}\right)^{1\text{st}}=\frac{\alpha_m}{L}\,H_{0\varphi}(z)\left(1-j\,\frac{\omega}{2}\,\frac{\alpha_m}{L}\,\mu h_{0\varphi}^2\int_{\text{slot}}e^{-jk_0|z-\xi|}d\xi+j\,\frac{\omega}{2}\,\frac{\alpha_e}{cL}\,h_{0\varphi}e_{0r}\int_{\text{slot}}\text{sgn}(\xi-z)e^{-jk_0|z-\xi|}dz\xi\right), \tag{8a}$$

$$\left(\frac{dP_r}{dz}\right)^{1\text{st}}=\frac{\varepsilon\alpha_e}{L}\,E_{0r}(z)\left(1-j\,\frac{\omega}{2}\,\frac{\varepsilon\alpha_e}{L}\,e_{0r}^2\int_{\text{slot}}e^{-jk_0|z-\xi|}d\xi+j\,\frac{\omega}{2}\,\frac{\alpha_m}{cL}\,h_{0\varphi}e_{0r}\int_{\text{slot}}\text{sgn}(\xi-z)e^{-jk_0|z-\xi|}d\xi\right) \tag{8b}$$

from which we derive the first order approximation for the longitudinal impedance

$$Z(\omega)=\frac{Z_0k_0^2}{32\pi^3b^4\ln(d/b)}\left[(\alpha_m+\alpha_e)^2+(\alpha_m-\alpha_e)^2\,\frac{1-\cos(2k_0L)}{2k_0^2L^2}\right]+j\,\frac{Z_0k_0}{4\pi^2b^2}\left\{(\alpha_m+\alpha_e)-\frac{(\alpha_m-\alpha_e)^2}{8\pi b^2\ln(d/b)L}\left[1-\frac{\sin(2k_0L)}{2k_0L}\right]\right\}. \tag{9}$$

Figures 2 and 3 show the frequency behavior of the real and imaginary parts of the impedance for a long slot. It can be observed that the resonance effect, due to the slot length, is predominant in the real impedance, whereas it is negligible in the imaginary impedance. Similar behaviors were recently obtained by Fedotov and Gluckstern using a different method [9].

For a Gaussian bunch of length σ_z applying the standard definition of loss factor

$$k(\sigma_z)=\frac{1}{\pi}\int_0^\infty\text{Re}[Z(\omega)e^{-\omega^2\sigma_z^2/c^2}d\omega] \tag{10}$$

we obtain

$$k_l(\sigma_z)=\frac{Z_0c\sqrt{\pi}}{128\pi^4b^4\ln(d/b)\sigma_z}\times\left\{\frac{(\alpha_m+\alpha_e)^2}{\sigma_z^2}+\frac{(\alpha_m-\alpha_e)^2}{L^2}\,[1-e^{-(L/\sigma_z)^2}]\right\}. \tag{11}$$

In Fig. 4 we show the typical behavior of the loss factor for a rectangular slot (solid line) and compare it to the low

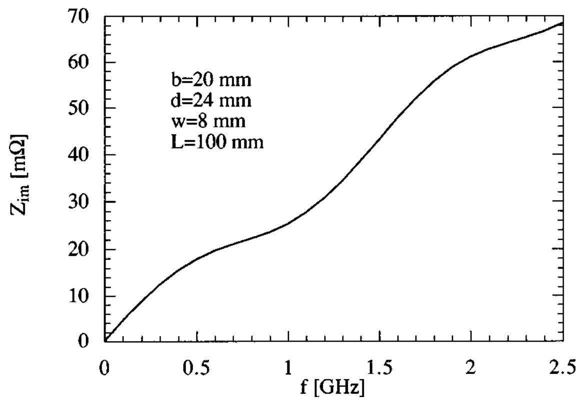

FIG. 2. Imaginary impedance of a rectangular slot.

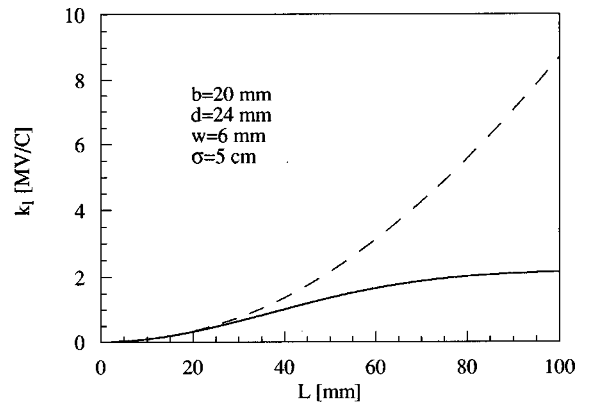

FIG. 4. Loss factor vs length for a rectangular slot (solid line). The dashed line is the low frequency approximation.

frequency expression [Eq. (13), dashed line]. One can see that the curves are almost identical for $L<\sigma_z/2$.

IV. ASYMPTOTIC BEHAVIOR

A. Limit for small apertures

In the limit of small apertures ($k_0L\ll1$) Eq. (9) yields the well known expressions for the imaginary and real parts of the impedance

$$Z_{\rm Im}(\omega)=\frac{Z_0k_0}{4\pi^2b^2}\,(\alpha_m+\alpha_e), \tag{12a}$$

$$Z_{\rm Re}(\omega)=\frac{Z_0k_0^2}{16\pi^3b^4\ln(d/b)}\,(\alpha_m^2+\alpha_e^2), \tag{12b}$$

which can be obtained independently by the modified Bethe theory [2] or by a field matching method [9]. Moreover, the loss factor for a small hole,

$$k_l(\sigma_z)=\frac{Z_0c\sqrt{\pi}}{64\pi^4b^4\ln(d/b)\sigma_z^3}\,(\alpha_m^2+\alpha_e^2), \tag{13}$$

can be easily obtained from Eq. (11) if the condition $L/\sigma_z \ll1$ is satisfied.

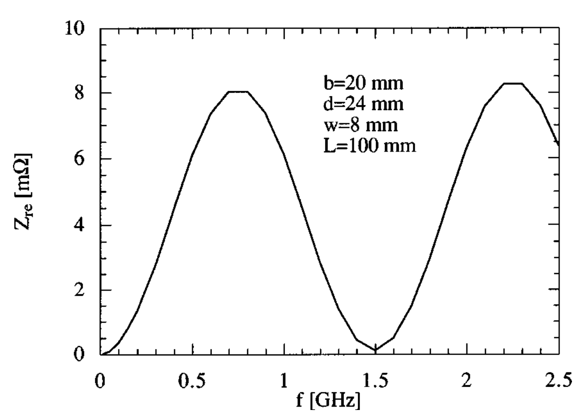

FIG. 3. Real impedance of a rectangular slot.

B. Limit for long slots

When the wavelength is much shorter than the slot length, from Eq. (9) we see that, considering $k_0L\gg1$, the impedance is given by

$$Z_{\rm IM}(\omega)\approx\frac{Z_0k_0}{4\pi^2b^2}\left[(\alpha_m+\alpha_e)-\frac{(\alpha_m-\alpha_e)^2}{8\pi b^2\ln(d/b)L}\right], \tag{14a}$$

$$Z_{\rm RE}(\omega)=\frac{Z_0k_0^2}{32\pi^3b^4\ln(d/b)}\,(\alpha_m+\alpha_e)^2. \tag{14b}$$

For very long slots we may use in Eqs. (14) the following relations [10]

$$\alpha_m=-\alpha_e=\alpha\propto L, \tag{15}$$

which show that, in this case, both real and imaginary parts of the impedance have their limit equal to zero. It must be stressed that this result is not applicable to the low frequency impedance since it does not fulfill the condition $k_0L\gg1$.

The loss factor can be obtained directly from Eq. (11) and it saturates for $L>2\sigma_z$. This effect is in agreement with [4]. In order to get the saturation value, we may use Eqs. (15) getting

$$k_l(\sigma_z)=\frac{Z_0c\sqrt{\pi}}{128\pi^4b^4\ln(d/b)\sigma_z}\left(\frac{2\alpha}{L}\right)^2. \tag{16}$$

It is worth noting that Eq. (16) is independent from the shape of the slot ends since the limit value of α is the same $(\pi W^2L/16)$.

V. COMPARISONS WITH NUMERICAL RESULTS

We have performed simulations with the numerical code MAFIA [11] in the case of both rectangular and rounded end slots of different length L, width W, and for a beam pipe thickness in the range $T=1-8$ mm (see Figs. 5–7). To this end, it has been necessary to slightly modify the equations to account for the wall thickness that changes the problem geometry and introduces attenuation for the field in the slot. Calling b_1 and b_2, respectively the inner and the outer radius of the beam pipe, one can see that the factor b^4 in the de-

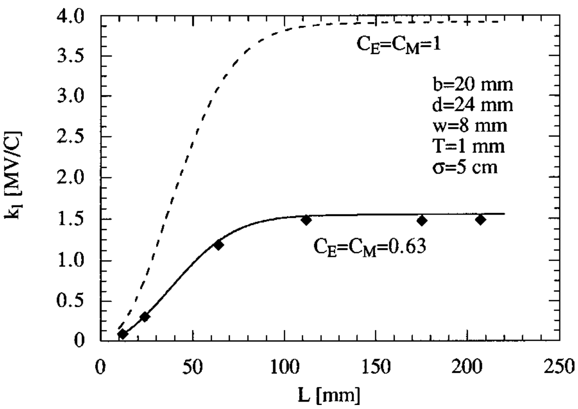

FIG. 5. Loss factor of a rectangular slot 8 mm wide. Black diamonds are MAFIA points.

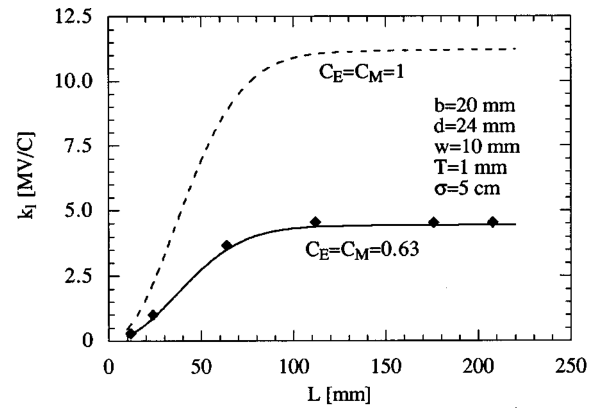

FIG. 6. Loss factor of a rectangular slot 10 mm wide. Black diamonds are MAFIA points.

nominator of Eqs. (9) and (11) has to be replaced by the product $b_1^2 b_2^2$. Furthermore the polarizabilities must be corrected; in a thin wall their expressions are [10]

$$\alpha_e = -\frac{\pi}{16} W^2 L(1-0.5663x+0.1398x^2), \qquad (17a)$$

$$\alpha_m = \frac{\pi}{16} W^2 L(1+0.3577x-0.0356x^2) \qquad (17b)$$

for rectangular slots, while for rounded end they become

$$\alpha_e = -\frac{\pi}{16} W^2 L(1-0.7650x+0.1894x^2), \qquad (18a)$$

$$\alpha_m = \frac{\pi}{16} W^2 L(1-0.0857x-0.0654x^2), \qquad (18b)$$

where $x = W/L$.

In general, the effect of the pipe thickness can be considered, using the approximation developed by McDonald [12], as a function of the corresponding ''zero-thickness'' polarizabilities as

$$\tilde{\alpha}_e = C_E \alpha_e e^{-\gamma T} \quad \text{with} \quad \gamma = \pi\sqrt{1/L^2 + 1/W^2} \qquad (19a)$$

$$\tilde{\alpha}_m = C_M \alpha_m e^{-\pi T/W}, \qquad (19b)$$

where T is the wall thickness and C_E and C_M are constants to be determined.

This approach has been successfully applied for a circular hole where the value of the constants' analog to C_E and C_M are numerically calculated in [12] and found to be approximately equal to 0.83. In our case, it is questionable whether this approach may be applicable to the case of T/W ratios as small as 0.1. In [12] it is suggested that Eqs. (19) are very accurate, in the case of circular holes, when the thickness-diameter ratio $T/D > 0.25$, though numerical results in that paper show that, for $T/D \approx 0.1$, the error in the thick-wall polarizability determination is around 10%. A comparison of the analytical (dashed lines in Figs. 5 and 6) and numerical results obtained with MAFIA (black diamonds), suggests the following values: $C_E = C_M = 0.63$ for $T/W = 0.1$ and 0.2. In order to check this result, the loss factor has been computed numerically in the saturation region versus the wall thickness. Comparison with expression (16) corrected by Eqs. (19) gives as best fit $C_E = C_M = 0.615$ (See Fig. 7).

It should be pointed out that the theory adopted to account for the slot thickness is, strictly speaking, valid only in the static approximation. In our case, the primary field $H_{0\varphi}$ field does not couple to the $TE_{n,0}$ modes of the rectangular slot, while the first coupling mode is the $TE_{0,1}$ having its cutoff frequency at $f_{[0,1]} = c/2W$. Thus the static approximation for the attenuation is still valid in the frequency range of our interest, namely, $\lambda \gg W$.

VI. CONCLUSIONS

In this paper we have obtained an approximated analytical expression for the coupling impedance and loss factor of a long slot in a coaxial pipe. When the slot is longer than the wavelength, the real part of the impedance presents a typical resonant behavior related to the slot length. Our results are in good agreement with those obtained in literature with different methods and with MAFIA simulations.

APPENDIX

From Eqs. (1) and (2)

$$dM_\varphi(z) = \frac{\alpha_m}{L}[H_{0\varphi}(z) - H_{s\varphi}(z)]dz, \qquad (A1)$$

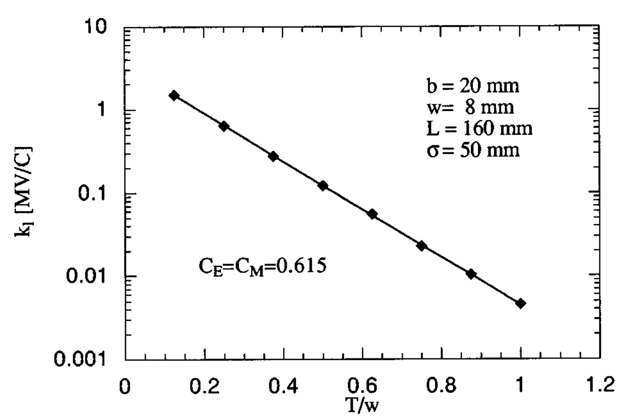

FIG. 7. Loss factor of a rectangular slot 8 mm wide in the saturation region ($L = 160$ mm) vs T/W ratio. Black diamonds are MAFIA points.

$$dP_r(z)=\frac{\varepsilon\alpha_e}{L}[E_{0r}(z)-E_{sr}(z)]dz, \tag{A2}$$

where the primary incident field is given by

$$E_{0r}(z)=Z_0H_{0\varphi}(z)=Z_0\frac{q}{2\pi b}e^{-jk_0z}. \tag{A3}$$

Each infinitesimal element of the slot at position ξ, is the source of a forward and a backward TEM wave with amplitude [5]

$$da(\xi)=\frac{j\omega}{2}[\mu h_{0\varphi}dM_\varphi(\xi)+e_{0r}dP_r(\xi)], \tag{A4}$$

$$db(\xi)=-\frac{j\omega}{2}[\mu h_{0\varphi}dM_\varphi(\xi)-e_{0r}dP_r(\xi)], \tag{A5}$$

where $h_{0\varphi}$ and e_{0r} are the TEM modal function.

The total scattered field is the integral along the slot of all these waves. We may write

$$H_{s\varphi}(z)=\int_{-L/2}^{z}h_{0\varphi}e^{-jk_0(z-\xi)}da(\xi)-\int_{z}^{L/2}h_{0\varphi}e^{jk_0(z-\xi)}db(\xi), \tag{A6}$$

$$E_{sr}(z)=\int_{-L/2}^{z}e_{0r}e^{-jk_0(z-\xi)}da(\xi)+\int_{z}^{L/2}e_{0r}e^{jk_0(z-\xi)}db(\xi) \tag{A7}$$

from which Eqs. (3) are easily derived.

[1] S. S. Kurennoy, Part. Accel. **39**, 1 (1992).
[2] S. De Santis, M. Migliorati, L. Palumbo, and M. Zobov, Phys. Rev. E **54**, 800 (1996).
[3] A. Fedotov and R. L. Gluckstern, Phys. Rev. E **54**, 1930 (1996).
[4] G. V. Stupakov, Phys. Rev. E **51**, 3515 (1995).
[5] R. E. Collin, *Field Theory of Guided Waves*, 2nd ed. (IEEE, New York, 1991).
[6] S. De Santis and L. Palumbo, Phys. Rev. E **55**, 2052 (1997).
[7] S. De Santis, A. Mostacci, and L. Palumbo, Phys. Rev. E **56**, 5990 (1997).
[8] L. Palumbo, V. G. Vaccaro, and M. Zobov, in *CAS Advanced Accelerator School* (CERN, Geneva, 1995), p. 331.
[9] A. Fedotov and R. L. Gluckstern, Phys. Rev. E **56**, 7217 (1997).
[10] S. S. Kurennoy, SSC Laboratory Report No. SSCL-636, 1993 (unpublished); N. A. McDonald, IEEE Trans. Microwave Theory Tech. **33**, 1146 (1985); **35**, 20 (1987).
[11] M. Bartsch *et al.*, Comput. Phys. Commun. **72**, 22 (1992).
[12] N. A. McDonald, IEEE Trans. Microwave Theory Tech. **20**, 689 (1972).